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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the presnt stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

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$$G_1 = p_0\, xy \left[\frac{3-\sigma}{20} h^2 + \frac{1}{12}(x^2+y^2) - \frac{1-\sigma}{96}(3x^2+5y^2)\right]$$

$$G_2 = p_0\, xy \left[\frac{3-\sigma}{20} h^2 + \frac{1}{12}(x^2+y^2) - \frac{1-\sigma}{96}(5x^2+3y^2)\right]$$

$$H_1 = -p_0 h^2 \left[\frac{3-\sigma}{280} h^2 - \frac{2-\sigma}{40}(x^2+y^2) - \frac{1}{20} h^2\right.$$

$$\left. - \frac{2}{3} p_0 \left[\frac{1-\sigma}{2^8}(5x^4+18x^2y^2+5y^4) + \frac{3}{16} h^2 (x^2+y^2)\right]\right]$$

Now, the five equations of equilibrium are :*

$$\frac{\partial T_1}{\partial x} + \frac{\partial S_1}{\partial y} + X' = 0; \quad \frac{\partial S_1}{\partial x} + \frac{\partial T_2}{\partial y} + Y' = 0$$

$$\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial y} + Z' = 0; \quad \frac{\partial H_1}{\partial x} - \frac{\partial G_2}{\partial y} + N_2 + L' = 0$$

$$\frac{\partial G_1}{\partial x} - \frac{\partial H_1}{\partial y} - N_1 + M' = 0.$$

These equations are satisfied by the expressions for the stress-resultants and stress couples that we have calculated, if X', Y', L' and M' all vanish, and Z' is equivalent to $-p_0 xy$.

From equation (42), we see that the middle-plane is bent into a surface given by the equation :

$$W = -\frac{1}{3}\frac{p_0}{D} xy \left[\frac{1}{128}(x^2+y^2)^2 - \frac{h^2}{8(1-\sigma)}(x^2+y^2) - \frac{h^4}{1-\sigma}\right] \quad \ldots \quad (52)$$

If we denote $\frac{\partial^2 W}{\partial x^2}$, $\frac{\partial^2 W}{\partial y^2}$ and $\frac{\partial^2 W}{\partial x \partial y}$ by K_1, K_2 and T respectively,

and if D denotes the flexural rigidity of the plate, we may easily show

* Love: *Elasticity* (Third Edition), p. 471.

195325

that

$$G_1 = -D(K_1 + \sigma K_2) + \frac{123 + 116.\sigma + \sigma^2}{20(1-\sigma)} p_0 h^2 xy \quad \ldots \quad (53)$$

$$G_2 = -D(K_2 + OK_1) + \frac{123 + 116.\sigma + \sigma^2}{20(1-\sigma)} p_0 h^2 xy \quad \ldots \quad (54)$$

$$H_1 = D(1-\sigma)\tau + \frac{2-\sigma}{40} p_0 h^2 (x^2 + y^2) - \frac{247 - 3\sigma}{840} p_0 h^4 \quad \ldots \quad (55)$$

Thus the formulæ

$$G_1 = -D(K_1 + \sigma K_2)$$

$$G_2 = -D(K_2 + \sigma K_1)$$

$$H_1 = D(1-\sigma)T$$

are approximately correct when the plate is thin, that is when h is small.

PART II.

Our object in this part is to adapt the results of the previous section to the case of a circular plate with clamped edge. This can be done by combining the solutions that we have obtained with those of articles 301 and 303 of Love's *Elasticity*. After combining the three solutions, our problem reduces to adjusting them in such a manner as to satisfy the boundary conditions of the given problem.

The stress-systems that occur in the articles referred to above, arise from the plane harmonic functions θ_0 and θ_1 which we have neglected in the expression for θ.

2. The displacement components due to θ_0 and θ_1 are given as follows:—

$$u = \frac{1}{E}\left(\xi + \frac{1}{2}\sigma z^2 \frac{\partial \theta_0}{\partial x}\right) - \frac{1+\sigma}{E}\frac{\partial \chi_0}{\partial x} \quad \ldots \quad (56)$$

$$v = \frac{1}{E}\left(\eta + \frac{1}{2}\sigma z^2 \frac{\partial \theta_0}{\partial y}\right) - \frac{1+\sigma}{E}\frac{\partial \chi_0}{\partial y} \quad \ldots \quad (57)$$

$$w = -\frac{\sigma}{E} z\theta_0 \quad \ldots \quad (58)$$

where ξ, η are a pair of conjugate functions of x and y satisfying the following relations

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = \theta_0 \text{ and } \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x}, \qquad \dots (59)$$

and χ_0 in its most general form is expressed by

$$\chi_0 = \tfrac{1}{2} x \xi + f \qquad \dots (60)$$

where f is a plane harmonic function.

The part of the displacement components arising out of θ_1 are given by:

$$u = -\frac{1}{E}\left[(1+\sigma)\frac{\partial \chi_1'}{\partial x} + \frac{1}{6}(2-\sigma) z^3 \frac{\partial \theta_1}{\partial x}\right] \qquad \dots (61)$$

$$v = -\frac{1}{E}\left[(1+\sigma)\frac{\partial \chi_1'}{\partial y} + \frac{1}{6}(2-\sigma) z^3 \frac{\partial \theta_1}{\partial y}\right] \qquad \dots (62)$$

$$w = \frac{1}{E}\left[(1+\sigma)\chi_1' + (h^2 - \tfrac{1}{2}\sigma z^2)\theta_1\right] \qquad \dots (63)$$

where χ_1' in its most general form is expressed by the relation:

$$\chi_1' = -\frac{1-\sigma}{2(1+\sigma)} x \xi_1 + F_1 \qquad \dots (64)$$

where F_1 is a plane harmonic function, and ξ_1 and η_1 are a pair of conjugate functions of x, y given by the relations:

$$\frac{\partial \xi_1}{\partial x} = \frac{\partial \eta_1}{\partial y} = \theta_1 \text{ and } \frac{\partial \xi_1}{\partial y} = -\frac{\partial \eta_1}{\partial x}. \qquad \dots (65)$$

When the edge is clamped, the boundary conditions holding at the edge-lines are:

$$U = 0,\ V = 0,\ W = 0 \text{ and } \frac{\partial W}{\partial \nu} = 0$$

where U, V, W denote the displacement components of any point on the middle-plane of the plate ($z = 0$), and ν denotes the direction of the normal to the edge-line.

Let us now try to fulfil these conditions by combining the results already referred to and the results we have established in Part I.

This displacement components of any point of the middle-plane of the plate arising out of Θ_0 are given by

$$U = \frac{1}{E}\left[\xi - (1+\sigma)\frac{\partial \chi_0}{\partial x}\right] \quad \ldots \quad (66)$$

$$V = \frac{1}{E}\left[\eta - (1+\sigma)\frac{\partial \chi_0}{\partial y}\right] \quad \ldots \quad (67)$$

$$W = 0 \quad \ldots \quad (68)$$

and those arising out of Θ_1 are given by

$$U = 0, \quad \ldots \quad (69)$$

$$V = 0, \quad \ldots \quad (70)$$

$$W = \frac{1}{E}\left[(1+\sigma)\chi_1' + h^2\Theta_1\right]; \quad \ldots \quad (71)$$

and those arising out of the expression that we have calculated are given by:—

$$U = \frac{1+\sigma}{8E} p_0 x^2 y - \frac{1+\sigma}{8E} p_0 y^3 \quad \ldots \quad (72)$$

$$V = \frac{1+\sigma}{8E} p_0 x y^2 - \frac{1+\sigma}{8E} p_0 x^3 \quad \ldots \quad (73)$$

$$W = -\frac{1+\sigma}{2E} p_0 h^{-3} xy \left[\frac{1-\sigma}{2^7}(x^2+y^2)^2 - \frac{h^2}{2^3}(x^2+y^2) - h^4\right] \quad \ldots \quad (74)$$

Combining these together, we get for the complete expressions for the displacement components of any point in the middle-plane

$$U = \frac{1}{E}\left[\xi - (1+\sigma)\frac{\partial \chi_0}{\partial x} + \frac{1}{8}(1+\sigma)p_0 x^2 y - \frac{1}{8}(1+\sigma)p_0 y^3\right] \quad \ldots \quad (75)$$

$$V = \frac{1}{E}\left[\eta - (1+\sigma)\frac{\partial \chi_0}{\partial y} + \frac{1}{8}(1+\sigma)p_0 x y^2 - \frac{1}{8}(1+\sigma)p_0 x^3\right] \quad \ldots \quad (76)$$

$$W = \frac{1}{E}\left[(1+\sigma)\chi_1' + h^2\Theta_1 - \frac{1}{2^8}\frac{1-\sigma^2}{h^3} p_0 xy (x^2+y^2)^2 + \frac{1}{16}\frac{1+\sigma}{h} p_0 xy (x^2+y^2) + \frac{1}{2}(1+\sigma)p_0 h xy.\right] \quad \ldots \quad (77)$$

Now, let

$$\xi = \alpha\,(y^3 - 3x^2 y) \qquad \dots \quad (78)$$

where α is a constant.

Then $$\eta = \alpha\,(x^3 - 3xy^2). \qquad \dots \quad (79)$$

Now $$\chi_0 = \tfrac{1}{2}\, x\,\xi + f.$$

Let us take

$$f = \beta xy + \alpha xy\,(x^2 - y^2) \qquad \dots \quad (80)$$

where β is another constant.

Then $$\chi_0 = -\tfrac{1}{2}\,\alpha\, xy\,(x^2 + y^2) + \beta\, xy \qquad \dots \quad (81)$$

Hence $$\frac{\partial \chi_0}{\partial x} = -\frac{1}{2}\,\alpha\,(3x^2 y + y^3) + \beta y \qquad \dots \quad (82)$$

$$\frac{\partial \chi_0}{\partial y} = -\frac{1}{2}\,\alpha\,(x^3 + 3xy^2) + \beta x \qquad \dots \quad (83)$$

With these values of ξ, η and χ_0 we have

$$U = \frac{1}{E}\Big[-(1+\sigma)\,\beta y + x^2 y\left\{-3\alpha + \frac{3}{2}\,\alpha\,(1+\sigma) + \frac{1}{8}\,(1+\sigma)\,p_0\right\}$$
$$+ y^3\left\{\alpha + \frac{1}{2}\,\alpha\,(1+\sigma) - \frac{1}{8}\,(1+\sigma)\,p_0\right\}\Big] \qquad \dots \quad (84)$$

$$V = \frac{1}{E}\Big[-(1+\sigma)\,\beta x + x^3\left\{\alpha + \frac{1}{2}\,\alpha\,(1+\sigma) - \frac{1}{8}\,(1+\sigma)\,p_0\right\}$$
$$+ xy^2\left\{-3\alpha + \frac{3}{2}\,\alpha\,(1+\sigma) + \frac{1}{8}\,(1+\sigma)\,p_0\right\}\Big] \dots \quad (85)$$

Now, if we take

$$\alpha = \frac{1}{4}\cdot\frac{1+\sigma}{3-\sigma}\,p_0 \quad\text{and}\quad \beta = \frac{1}{4}\cdot\frac{\sigma}{3-\sigma}\,p_0\,a^2.$$

We may easily shew that the above expressions become

$$U = -\frac{1+\sigma}{4E}\cdot\frac{\sigma\, p_0 y}{3-\sigma}\,(a^2 - r^2) \qquad \dots \quad (86)$$

$$V = -\frac{1+\sigma}{4E}\cdot\frac{\sigma\, p_0 x}{3-\sigma}\,(a^2 - r^2) \qquad \dots \quad (87)$$

These expressions vanish over the edge line $r = a$.

It may also be easily shewn that there is only one way of choosing the values of θ_0, χ_0, ξ and η satisfying the given boundary conditions.

Now θ_0 is given by

$$\theta_0 = \frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}.$$

Hence

$$\theta_0 = -6\alpha xy = -\frac{3}{2}\frac{1+\sigma}{3-\sigma} p_0 xy \qquad \dots \quad (88)$$

We have seen that W is given by

$$W = \frac{1}{E}\left[(1+\sigma)\chi_1' + h^2\theta_1 - \frac{1}{2^8}\frac{1-\sigma^2}{h^3} p_0 xy (x^2+y^2)^2 \right.$$

$$\left. + \frac{1}{16}\frac{1+\sigma}{h} p_0 xy (x^2+y^2) + \frac{1}{2}(1+\sigma) p_0 h xy \right]$$

where θ_1 is a plane harmonic function and χ_1' satisfies the equation

$$\nabla_1^2 \chi_1' = -\frac{1-\sigma}{1+\sigma}\theta_1.$$

$$\therefore \qquad \nabla_1^4 \chi_1' = 0.$$

From these we easily see that W satisfies the equation

$$D\nabla_1^4 W = -p_0 xy \qquad \dots \quad (89)$$

To determine W, we have therefore the equation (89) and the boundary conditions $W = 0$ and $\frac{\partial W}{\partial \nu} = 0$ at the edge-line.

There is only one value of W that will satisfy all these conditions and the expression for W is given by :

$$W = -\frac{p_0}{312 D} xy (a^2 - r^2)^2. \qquad \dots \quad (90)$$

When W is determined, θ_1 is given by the equation

$$\nabla_1^2 W = -\frac{1-\sigma}{E}\theta_1 - \frac{1-\sigma^2}{Eh^3} p_0 xy \left[\frac{1}{8}(x^2+y^2) - \frac{3}{4}\frac{h^2}{1-\sigma}\right]$$

$$= -\frac{1-\sigma}{E}\theta_1 - \frac{1}{3}\frac{p_0}{D} xy \left[\frac{1}{4}(x^2+y^2) - \frac{3}{2}\frac{h^2}{1-\sigma}\right] \qquad \dots \quad (91)$$

Also from (90), we get

$$\nabla_1^2 W = -\frac{4}{39}\frac{p_0}{D}xy\left(x^2+y^2-\frac{3}{4}a^2\right) \qquad \ldots \quad (92)$$

After a little simplification, we easily get

$$\theta_1 = \frac{p_0 E}{D}xy\left[\frac{1}{42}\frac{x^2+y^2}{1-\sigma}-\frac{1}{13}\frac{a^2}{1-\sigma}+\frac{1}{2}\frac{h^2}{(1-\sigma)^2}\right] \ldots \quad (93)$$

And it is easy to shew that

$$\chi_1' = -\frac{p_0}{312D}\cdot\frac{E}{1+\sigma}xy(a^2-r^2)^2-\frac{p_0}{D}\cdot\frac{Eh^2}{1+\sigma}xy\left[\frac{1}{42}\frac{x^2+y^2}{1-\sigma}\right.$$

$$\left.-\frac{1}{13}\frac{a^2}{1-\sigma}+\frac{1}{2}\frac{h^2}{(1-\sigma)^2}\right]+\frac{1}{2^8}\frac{1-\sigma}{h^3}p_0 xy(x^2+y^2)z$$

$$-\frac{1}{16}\frac{p_0}{h}xy(x^2+y^2)+\frac{1}{2}p_0 xy. \qquad \ldots \quad (94)$$

The values of θ_1, χ'_1, ξ, η, χ_0 and θ_0 being known, the general expressions for the displacement components at any point may be obtained by substituting their values in

$$u = \frac{1}{E}\left[\xi+\frac{1}{2}\sigma z^2\frac{\partial\theta_0}{\partial x}\right]-\frac{1+\sigma}{E}\frac{\partial\chi_0}{\partial x}$$

$$-\frac{1}{E}\left[(1+\sigma)\frac{\partial\chi'_1}{\partial\chi}+\frac{2-\sigma}{\sigma}z^3\frac{\partial\theta_1}{\partial x}\right]$$

$$+\frac{1+\sigma}{E}\frac{p_0}{h^3}y\left[\frac{1}{8}h^3x^2+\frac{3}{16}h^2x^2z+\frac{1-\sigma}{2^8}x^2z(5x^2+6y^2)\right.$$

$$-\frac{2-\sigma}{16}x^2z^3+\frac{3-\sigma}{80}z^5-\frac{2-\sigma}{48}y^2z^3$$

$$-\frac{1}{8}h^3y^2+\frac{1}{16}h^2y^2z-\frac{1}{8}h^2z^3$$

$$\left.+\frac{1-\sigma}{2^8}y^4z+\frac{1}{4}h^3z^2\right] \qquad \ldots \quad (95)$$

The expression for v is obtained from (95) by interchanging x and y, ξ and η, on the right,

and

$$w = -\frac{\sigma}{E} z\,\Theta_0 + \frac{1}{E}\left[(1+\sigma)\,\chi'_1 + \left(h^2 - \frac{1}{2}\sigma z^2\right)\Theta_1\right]$$

$$+ \frac{1+\sigma}{E}\,p_0 h^{-3} xy\left[-\frac{1}{2}h^3 z + \frac{\sigma}{16}(x^2+y^2)\,z^2\right.$$

$$+ \frac{1+\sigma}{16}z^4 + \frac{1}{2}h^4 + \frac{1}{16}h^2(x^2+y^2)$$

$$\left. - \frac{1-\sigma}{2^8}(x^2+y^2)^2\right] \quad \ldots \quad (96)$$

Thus the problem is completely solved.

ON TIDAL WAVES IN AN ELLIPTIC CANAL OF VARIABLE DEPTH

By A. C. Banerji, M.A. (Cantab.)

and

Rama Shankar Varma, M.Sc (Allahabad),
Allahabad University.

The object of this paper is to consider the problem of Tidal waves in a canal where the banks have the form of similar elliptic arcs whose eccentricity differs from unity by a small quantity. The depth is taken to vary along the length of the canal in the form of an elliptic arc which has also its eccentricity differing from unity by a small quantity. It is believed that this has not been investigated before.

If η be the elevation of liquid, b and h the breadth, and the mean depth over the width b, respectively, the equation of motion is

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{g}{b}\frac{\partial}{\partial x}\left(bh\frac{\partial \eta}{\partial x}\right),$$

where b and h are functions of x. In the case of simple harmonic motion, where $\eta \propto \cos(\sigma t + \varepsilon)$, the above equation becomes

$$\frac{g}{b}\frac{\partial}{\partial x}\left(bh\frac{\partial \eta}{\partial x}\right) + \sigma^2 \eta = 0^* \qquad \ldots \quad (1)$$

Let us take

$$b = b_0 \sqrt{1 - q_1^2 x^2} \text{ and } h = h_0 \sqrt{1 - q_2^2 x^2},$$

where q_1 and q_2 are small.

The equation (1) then becomes

$$\frac{g}{b_0\sqrt{1-q_1^2x^2}}\frac{\partial}{\partial x}\left\{b_0\sqrt{1-q_1^2x^2}\,.\,h_0\sqrt{1-q_2^2x^2}\,\frac{\partial \eta}{\partial x}\right\} + \sigma^2\eta = 0 \ldots \quad (2)$$

We shall take q_1 and q_2 to be so small that we may neglect sixth and higher orders of small quantities in q_1 and q_2.

* Lamb: *Hydrodynamics* (5th Edition), Art. 186.

The equation (2) becomes

$$\frac{\partial^2\eta}{\partial x^2}\left\{1-\frac{x^2}{2}\left(q_1^2+q_2^2\right)-\frac{x^4}{8}\left(q_1^2-q_2^2\right)\right\}^2$$

$$-\frac{\partial\eta}{\partial x}\left\{x\left(q_1^2+q_2^2\right)+\frac{x^3}{2}\left(q_1^2-q_2^2\right)^2\right\}$$

$$+k^2\left(1-\frac{1}{2}x^2q_1^2-\frac{1}{8}x^4q_1^4\right)\eta=0,$$

where
$$k^2=\frac{\sigma^2}{gh_0}.$$

This further reduces to

$$\frac{\partial^2\eta}{\partial x^2}-\frac{\partial\eta}{\partial x}\left\{x(q_1^2+q_2^2)+x^3(q_1^4+q_2^4)\right\}$$

$$+k^2\left(1+\frac{1}{2}x^2q_2^2+\frac{3}{8}x^4q_2^4\right)\eta=0 \qquad \dots (3)$$

The equation (3) can be written as

$$\frac{\partial^2\eta}{\partial x^2}+P\frac{\partial\eta}{\partial x}+Q=0,$$

where

$$P=-\left\{x(q_1^2+q_2^2)+x^3(q_1^4+q_1^4)\right\}$$

$$Q=k^2\left\{1+\frac{1}{2}x^2q_2^2+\frac{3}{8}x^4q_2^4\right\}$$

We shall now change the dependent variable from η to u, by the transformation,

$$\eta=e^{-\frac{1}{2}\int P dx}.u.$$

Thus we have

$$\frac{\partial^2 u}{\partial x^2}+Iu=0 \qquad \dots (4)$$

where

$$I=Q-\frac{1}{2}\frac{\partial P}{\partial x}-\frac{1}{4}P^2$$

$$=k^2\left(1+\frac{1}{2}x^2q_2^2+\frac{3}{8}x^4q_2^4\right)+\frac{1}{2}\left(q_1^2+q_2^2\right)$$

$$+\frac{3}{2}x^2(q_1^4+q_2^4)-\frac{1}{4}x^2(q_1^2+q_2^2)^2,$$

where sixth and higher orders of small quantities in q_1 and q_2 have been neglected.

Re-arranging, we get

$$I = k^2 + \frac{1}{2} q_1^2 + \frac{1}{2} q_2^2 (1 + k^2 x^2) + \frac{5}{4} q_1^4 x^2$$

$$+ q_2^4 \left(\frac{3}{8} k^2 x^4 + \frac{5}{4} x^2 \right) - \frac{1}{2} q_1^2 q_2^2 x^2.$$

Following the method of Priestley* and Dhar†, we shall write (4) in the form

$$\frac{\partial^2 u}{\partial x^2} + k^2 u = \left\{ -\frac{1}{2} q_1^2 - \frac{1}{2} q_2^2 (1 + k^2 x^2) - \frac{5}{4} q_1^4 x^2 \right.$$

$$\left. - q_2^4 \left(\frac{3}{8} k^2 x^4 + \frac{5}{4} x^2 \right) + \frac{1}{2} q_1^2 q_2^2 x^2 \right\} u$$

$$= Ru, \text{ say} \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

We know that cos kx and sin kx are any two independent solutions of

$$\frac{\partial^2 u}{\partial x^2} + k^2 u = 0 \quad \ldots \quad \ldots \quad (6)$$

Let us now assume that

$$u = u_1 \cos kx + u_2 \sin kx \quad \ldots \quad \ldots \quad (7)$$

is a general solution of (5), where u_1 and u_2 are two arbitrary functions of x, whose forms are to be obtained subject to the condition that

$$\cos kx . \frac{du_1}{dx} + \sin kx . \frac{du_2}{dx} = 0. \quad \ldots \quad \ldots \quad (8)$$

Substituting u, as given by (7) in equation (5) and simplifying by the help of (8), we obtain

$$-k \frac{du_1}{dx} \sin kx + k \frac{du_2}{dx} \cos kx = R (u_1 \cos kx + u_2 \sin kx) \quad \ldots \quad (9)$$

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* Priestley : *Proc. Lond. Math. Soc.* (2), Vol. XX (1922), page 38.

† S. C. Dhar : *Bull. Calcutta Math. Soc.*, Vol. XVIII (1927), page 111.

From (8) we get

$$-\frac{\dfrac{du_1}{dx}}{\sin kx} = \frac{\dfrac{du_2}{dx}}{\cos kx} \quad \dots \quad (10)$$

Putting each of these equal to X, we obtain from (9)

$$kX = R\,(u_1 \cos kx + u_2 \sin kx). \quad \dots \quad (11)$$

We have from (7), (10), and (11)

$$\frac{du_1}{dx} = -\frac{R}{k}\, u \sin kx$$

$$\frac{du_2}{dx} = \frac{R}{k}\, u \cos kx.$$

These give

$$u_1 = A - \frac{1}{k}\int_0^x R\,(t)\, u\,(t) \sin kt\, dt$$

and

$$u_2 = B + \frac{1}{k}\int_0^x R\,(t)\, u\,(t) \cos kt\, dt,$$

where A and B are arbitrary constants.

More generally the lower limit of the two integrals should have been α, where α is an arbitrary constant; but as the values of these integrals within the limits 0 and α will themselves be arbitrary constants, they can be included in the arbitrary constants A and B; so we are justified in taking the lower limits to be zero.

Hence we find that the solution of (5) also satisfies the Volterra equation

$$u\,(x) = (A \cos kx + B \sin kx)$$
$$+ \frac{1}{k}\int_0^x R\,(t)\, u\,(t)\, \{ \sin kx \cos kt - \sin kt \cos kx \}\, dt \quad \dots \quad (12)$$

We find that R is an even function of x.

The integral equation

$$u\,(x) = B \sin kx$$
$$+ \frac{1}{k}\int_0^x R\,(t)\, u\,(t)\, \{ \sin kx \cos kt - \sin kt \cos kx \}\, dt \quad \dots \quad (13)$$

will give odd solutions of (5); for by writing $-x$ for x in (13), we get

$$u(-x) = -\text{B} \sin kx$$
$$-\frac{1}{k}\int_0^{-x} \text{R}(t)\, u(t) \{ \sin kx \cos kt + \sin kt \cos kx \}\, dt$$

By changing the variable from t to $-t$, we get

$$-u(-x) = \text{B} \sin kx$$
$$+\frac{1}{k}\int^{x} \text{R}(t) \{ -u(-t) \} \{ \sin kx \cos kt - \sin kt \cos kx \}\, dt$$

It is known that the solution of an integral equation of the second kind is unique; and we find that $u(x)$ and $-u(-x)$ satisfy the same integral equation, hence

$$u(x) = -u(-x).$$

Therefore $u(x)$ given by (13) gives odd solutions of (5).

Similarly it can be shown that the even solutions of (5) are given by

$$u(x) = \text{A} \cos kx + \frac{1}{k}\int_0^{x} \text{R}(t)\, u(t) \{ \sin kx \cos kt - \sin kt \cos kx \}\, dt \qquad \dots \quad (14)$$

We shall first find the *odd* solutions.

Substituting the value of $\text{R}(t)$ in (13), we have

$$u(x) = \text{B} \sin kx + \frac{1}{k}\int_0^{x} \Big\{ \frac{1}{2} q_1^2 + \frac{1}{2} q_2^2 (1 + k^2t^2) + \frac{5}{4} q_1^4 t^2$$
$$+ q_2^4 \left(\frac{3}{8} k_2 t^4 + \frac{5}{4} t^2\right) - \frac{1}{2} q_1^2 q_2^2 t^2 \Big\} \sin k(t - x)\, u(t)\, dt$$

Let us assume

$$u(x) = \text{B} \{ f_0(x) + q_1^2 f_1(x) + q_2^2 f_2(x) + q_1^4 \psi_1(x) +$$
$$q_2^4 \psi_2(x) + q_1^2 q_2^2 f_{12}(x) \}$$

Neglecting small quantities of the sixth and higher orders in q_1 and q_2, and equating the different powers and product of q_1^2 and q_2^2, we have :

$$f_0(x) = \sin kx$$

$$f_1(x) = \frac{1}{2k}\int_0^{x} f_0(t) \sin k(t - x)\, dt$$

$$= \frac{1}{2k} \int_0^x \sin kt \sin k(t-x)\, dt$$

$$= -\frac{1}{4k^3} \sin kx + \frac{x}{4k} \cos kx$$

$$f_2(x) = \frac{1}{2k} \int_0^x (1+k^2t^2) f_0(t) \sin k(t-x)\, dt$$

$$= \frac{1}{2k} \int_0^x (1+k^2t^2) \sin kt \sin k(t-x)\, dt$$

$$= -\left(\frac{x^2}{8} + \frac{1}{8k^3}\right) \sin kx + \left(\frac{kx^3}{12} + \frac{x}{8k}\right) \cos kx .$$

$$\psi_1(x) = \frac{5}{4k} \int_0^x t^2 f_0(t) \sin k(t-x)\, dt + \frac{1}{2k} \int_0^x f_1(t) \sin k(t-x)\, dt$$

$$= \frac{5}{4k} \int_0^x t^2 \sin kt \sin k(t-x)\, dt$$

$$+ \frac{1}{2k} \int_0^x \left(-\frac{1}{4k^3} \sin kt + \frac{t}{4k} \cos kt\right) \sin k(t-x)\, dt$$

$$= -\left(\frac{3x^2}{32k^3} - \frac{13}{32k^4}\right) \sin kx + \left(\frac{5x^3}{24k} - \frac{13x}{32k^2}\right) \cos kx$$

$$\psi_2(x) = \frac{1}{k} \int_0^x \left(\frac{3}{8} k^2t^4 + \frac{5}{4} t^2\right) f_0(t) \sin k(t-x)\, dt$$

$$+ \frac{1}{2k} \int_0^x (1+k^2t^2) f_2(t) \sin k(t-x)\, dt$$

$$= \frac{1}{k} \int_0^x \left(\frac{3}{8} k^2 t^4 + \frac{5}{4} t^2\right) \sin kt \sin k(t-x)\, dt$$

$$+ \frac{1}{2k} \int_0^x (1+k^2t^2) \left\{ -\left(\frac{t^2}{8} + \frac{1}{8k^2}\right) \sin kt \right.$$

$$\left. + \left(\frac{kt^3}{12} + \frac{t}{8k}\right) \cos kt \right\} \sin k(t-x)\, dt$$

$$= -\left(\frac{k^2x^6}{288} + \frac{25x^4}{384} + \frac{13x^2}{128k^2} - \frac{17}{128k^4}\right) \sin kx$$

$$+ \left(\frac{kx^5}{48} + \frac{11x^3}{192k} - \frac{5x}{32k^3}\right) \cos kx$$

$$f_{12}(x) = -\frac{1}{2k}\int_0^x t^2 f_0(t)\sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x (1+k^2t^2) f_1(t)\sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x f_2(t)\sin k(t-x)\,dt$$

$$= -\frac{1}{2k}\int_0^x t^2 \sin kt \sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x (1+k^2t^2)\left(-\frac{1}{4k^2}\sin kt + \frac{t}{4k}\cos kt\right)\sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x \left\{-\left(\frac{t^2}{8}+\frac{1}{8k^2}\right)\sin kt\right.$$

$$\left. + \left(\frac{kt^3}{12}+\frac{t}{8k}\right)\cos kt\right\}\sin k(t-x)\,dt$$

$$= -\left(\frac{x^4}{48} - \frac{3x^2}{16k^2} + \frac{3}{32k^4}\right)\sin kx - \left(\frac{5x^3}{32k} - \frac{3x}{32k^3}\right)\cos kx.$$

Therefore, omitting the time-factor,

$$\eta = e^{-\frac{1}{2}\int P\,dx}\cdot u$$

$$= e^{\frac{x^2}{4}(q_1^2+q_2^2)+\frac{x^4}{8}(q_1^4+q_2^4)}\cdot u$$

Neglecting small quantities of sixth and higher orders, we have

$$\eta = B\left(1 + q_1^2\frac{x^2}{4} + q_2^2\frac{x^2}{4} + q_1^4\frac{5x^4}{32} + q_2^4\frac{5x^4}{32} + q_1^2q_2^2\frac{x^4}{16}\right)\{f_0(x) +$$

$$q_1^2 f_1(x) + q_2^2 f_2(x) + q_1^4 \psi_1(x) + q_2^4 \psi_2(x) + q_1^2 q_2^2 f_{12}(x)\}.$$

$$= B f_0(x) + B q_1^2\left\{\frac{x^2}{4} f_0(x) + f_1(x)\right\}$$

$$+ B q_2^2\left\{\frac{x^2}{4} f_0(x) + f_2(x)\right\}$$

$$+ B q_1^4\left\{\frac{5x^4}{32} f_0(x) + \frac{x^2}{4} f_1(x) + \psi_1(x)\right\}$$

$$+ B q_2^4\left\{\frac{5x^4}{32} f_0(x) + \frac{x^2}{4} f_2(x) + \psi_2(x)\right\}$$

$$+ B q_1^2 q_2^2\left\{\frac{x^4}{16} f_0(x) + \frac{x^2}{4} f_1(x) + \frac{x^2}{4} f_2(x) + f_{12}(x)\right.$$

Thus we get

$$\eta = B\sin kx + Bq_1^2\left\{\left(\frac{x^2}{4}-\frac{1}{4k^2}\right)\sin kx + \frac{x}{4k}\cos kx\right\}$$

$$+ Bq_2^2\left\{\left(\frac{x^2}{8}-\frac{1}{8k^2}\right)\sin kx + \left(\frac{kx^3}{12}+\frac{x}{8k}\right)\cos kx\right\}$$

$$+Bq_1^4\left\{\left(\frac{5x^4}{32}-\frac{5x^2}{32k^2}-\frac{13}{32k^4}\right)\sin kx + \left(\frac{13x^3}{48k}-\frac{13x}{32k^3}\right)\cos kx\right\}$$

$$+Bq_2^4\left\{-\left(\frac{k^2x^6}{288}-\frac{23x^4}{384}+\frac{17x^2}{128k^2}-\frac{17}{128k^4}\right)\sin kx\right.$$

$$\left.+\left(\frac{kx^5}{24}+\frac{17x^3}{192k}-\frac{5x}{32k^3}\right)\cos kx\right\}$$

$$+ Bq_1^2q_2^2\left\{\left(\frac{x^4}{96}+\frac{3x^2}{32k^2}-\frac{3}{32k^4}\right)\sin kx\right.$$

$$\left.+\left(\frac{kx^5}{48}-\frac{x^3}{16k}+\frac{3x}{32k^3}\right)\cos kx\right\}$$

$$= BF_1(x), \text{ say,}$$

where
$$k^2 = \frac{\sigma^2}{gh_0}.$$

Let us suppose that at its mouth ($x = a$), the canal communicates with the open sea in which a tidal oscillation $\eta = c\cos(\sigma t + \varepsilon)$ is maintained. Hence restoring the time-factor, the complete solution is

$$\eta = c\,\frac{F_1(x)}{F_1(a)}\cos(\sigma t + \varepsilon).$$

Let us now find the *even* solutions.

Substituting the value of R (t) in (14), we have

$$u(x) = A\cos kx + \frac{1}{k}\int_0^x\left\{\frac{1}{2}q_1^2 + \frac{1}{2}q_2^2(1+k^2t^2) + \frac{5}{4}q_1^4t^2\right.$$

$$\left. + q_2^4\left(\frac{3}{8}k^2t^4 + \frac{5}{4}t^2\right) - \frac{1}{2}q_1^2q_2^2t^2\right\}\sin k(t-x)\,u(t)\,dt.$$

Assume as before

$$u(x) = A\left\{\theta_0(x) + q_1^2\,\theta_1(x) + q_2^2\,\theta_2(x) + q_1^4\,\phi_1(x) + q_2^4\,\phi_2(x) + q_1^2q_2^2\,\theta_{12}(x)\right\},$$

Neglecting small quantities of sixth and higher orders in q_1 and q_2, and equating the different powers and product of q_1^2 and q_2^2, we have

$$\theta(x) = \cos kx;$$

$$\theta_1(x) = \frac{1}{2k}\int_0^x \theta_0(t)\sin k(t-x)\,dt$$

$$= \frac{1}{2k}\int_0^x \cos kt \sin t(t-k)\,dt$$

$$= -\frac{x}{4k}\sin kx;$$

$$\theta_2(x) = \frac{1}{2k}\int_0^x (1+k^2t^2)\,\theta_0(t)\sin k(t-x)\,dt$$

$$= \frac{1}{2k}\int_0^x (1+k^2t^2)\cos kt \sin k(t-x)\,dt$$

$$= -\frac{x^2}{8}\cos kx - \left(\frac{kx^3}{12}+\frac{x}{8k}\right)\sin kx;$$

$$\phi_1(x) = \frac{5}{4k}\int_0^x t^2\,\theta_0(t)\sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x \theta_1(t)\sin k(t-x)\,dt;$$

$$= \frac{5}{4k}\int_0^x t^2\cos kt \sin k(t-x)\,dt$$

$$-\frac{1}{2k}\int_0^x \frac{t}{4k}\sin kt \sin k(t-x)\,dt$$

$$= -\frac{11x^2}{32k^2}\cos kx - \left(\frac{5x^3}{24k}-\frac{11}{32k^3}\right)\sin kx;$$

$$\phi_2(x) = \frac{1}{k}\int_0^x \left(\frac{3}{8}k^2t^4+\frac{5}{4}t^2\right)\theta_0(t)\sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x (1+k^2t^2)\,\theta_2(t)\sin k(t-x)\,dt$$

$$= \frac{1}{k}\int_0^x \left(\frac{3}{8}k^2t^4+\frac{5}{4}t^2\right)\cos kt \sin k(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x\left(1+k^2t^2\right)\left\{-\frac{t^2}{8}\cos kt-\left(\frac{kt^3}{12}+\frac{t}{8k}\right)\sin kt\right\}.$$

$$\sin k\,(t-x)\,dt.$$

$$=-\left(\frac{k^2x^6}{288}+\frac{25x^4}{384}+\frac{15x^2}{128k^2}\right)\cos kx-\left(\frac{kx^5}{48}+\frac{13x^3}{192k}-\frac{15x}{128k^3}\right)\sin kx;$$

$$\theta_{12}(x)=-\frac{1}{2k}\int_0^x t^2\theta_0(t)\sin k\,(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x(1+k^2t^2)\,\theta_1(t)\sin k\,(t-x)\,dt$$

$$+\frac{1}{2k}\int_0^x\theta_2(t)\sin k\,(t-x)\,dt.$$

$$=-\frac{1}{2k}\int_0^x t^2\cos kt\sin k\,(t-x)\,dt$$

$$-\frac{1}{2k}\int_0^x(1+k^2t^2)\frac{t}{4k}\sin kt\sin k\,(t-x)\,dt$$

$$-\frac{1}{2k}\int_0^x\left\{\frac{t^2}{8}\cos kt+\left(\frac{kt^3}{12}+\frac{t}{8k}\right)\sin kt\right\}.\sin k\,(t-x)\,dt.$$

$$=-\left(\frac{x^4}{48}-\frac{5x^2}{32k^2}\right)\cos kx+\left(\frac{13x^3}{96k}-\frac{5}{32k^3}\right)\sin kx.$$

As before, omitting the time-factor and neglecting small quantities of sixth and higher orders, we have

$$\eta=A\,\theta_0(x)+A\,q_1^2\left\{\frac{x^2}{4}\theta_0(x)+\theta_1(x)\right\}+A\,q_2^2\left\{\frac{x^2}{4}\theta_0(x)+\theta_2(x)\right\}$$

$$+A\,q_1^4\left\{\frac{5x^4}{32}\theta_0(x)+\frac{x^2}{4}\theta_1(x)+\phi_1(x)\right\}$$

$$+A\,q_2^4\left\{\frac{5x^4}{32}\theta_0(x)+\frac{x^2}{4}\theta_2(x)+\phi_2(x)\right\}$$

$$+A\,q_1^2q_2^2\left\{\frac{x^4}{16}\theta_0(x)+\frac{x^2}{4}\theta_1(x)+\frac{x^2}{4}\theta_2(x)+\theta_{12}(x)\right\}$$

Thus we get

$$\eta = A\cos kx + Aq_1^2\left\{\frac{x^2}{4}\cos kx - \frac{x}{4k}\sin kx\right\}$$

$$+ A q_2^2\left\{\frac{x^2}{8}\cos kx - \left(\frac{kx^3}{12} + \frac{x}{8k}\right)\sin kx\right\}$$

$$+ A q_1^4\left\{\left(\frac{5x^4}{32} - \frac{11x^2}{32k^2}\right)\cos kx - \left(\frac{13x^3}{48k} - \frac{11}{32k^3}\right)\sin kx\right\}$$

$$+ Aq_2^4\left\{-\left(\frac{k^2x^6}{288} - \frac{23x^4}{384} + \frac{15x^2}{128k^2}\right)\cos kx\right.$$

$$\left. - \left(\frac{kx^5}{24} + \frac{19x^3}{192k} - \frac{15x}{128k^3}\right)\sin kx\right\}$$

$$+ Aq_1^2q_2^2\left\{\left(\frac{x^4}{96} + \frac{5x^2}{32k^2}\right)\cos kx - \left(\frac{kx^5}{48} - \frac{x^3}{24k} + \frac{5}{32k^3}\right)\sin kx\right.$$

$= AF_2(x)$, say,

where $$k^2 = \frac{\sigma^2}{gh_0}.$$

Let us suppose that at its mouth $(x = a)$, the canal communicates with the open sea in which a tidal oscillation $\eta = c\cos(\sigma t + \varepsilon)$ is maintained. Hence restoring the time-factor, the complete solution is

$$\eta = c\,\frac{F_2(x)}{F_2(a)}\cos(\sigma t + \varepsilon).$$

If we put $q_1 = 0$, we get the solution in the case of a canal of constant breadth, the depth curve being an ellipse. By putting $q_2 = 0$, we obtain the solution in the case of a canal of constant depth, the breadth curve being an ellipse. When both q_1 and q_2 are zero, we get the case of a canal of constant breadth and depth and the solutions clearly tally with those obtained otherwise.

In our investigations we assumed q_1 and q_2 to be so small that we neglected small quantities of sixth and higher orders in q_1 and q_2. The method is, however perfectly general and we can proceed to any degree of approximation we like, but the analysis becomes tedious.

ON THE FLOW OF HEAT IN A ROD COMPOSED OF TWO MATERIALS, DUE TO AN IMPULSIVE SOURCE.

BY T. SURYANARAYANA,
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In Heat conduction problems we require solutions of the equation

$$k\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} \quad \ldots \quad \ldots \quad (1)$$

which satisfy the initial and boundary conditions. Following Heaviside, we replace $\frac{\partial}{\partial t}$ by the operator $p\ (= kq^2)$ and regard p (or q) as obeying the laws of algebra. The equation (1) becomes

$$\frac{\partial^2 v}{\partial x^2} = q^2 v$$

of which the solutions are of the form

$$v = Ae^{qx} + Be^{-qx},$$

A and B being constants, *i.e.*, independent of x, to be determined from the conditions of the problem.

The expression thus obtained for v is the "operational solution", and it has to be interpreted according to the formulæ given by Heaviside, or by Bromwich's rule

$$f(q) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(\nu)\, e^{\lambda t}\frac{d\lambda}{\lambda}$$

where $$k\nu^2 = \lambda.$$

Heaviside developed these methods nearly forty years ago (see his *Electromagnetic Theory*, 3 Vols.) but interest in his methods has been revived after Bromwich's paper 'On Normal Co-ordinates in Dynamical Systems' in the *Proceedings of the London Mathematical Society* (1916) wherein he established the validity of Heaviside's methods. A systematic

account of Heaviside's method, with applications and references, has recently been published in the Cambridge Tract: '*Operational Methods in Mathematical Physics*' by Jeffreys.

We shall need two of the formulæ of interpretation:

$$\frac{F(p)}{\Delta(p)} = \frac{F(o)}{\Delta(o)} + \sum_{\alpha} \frac{F(\alpha)}{\Delta'(\alpha)} \frac{e^{\alpha t}}{\alpha} \quad \ldots \quad (2)$$

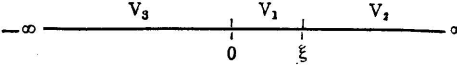
the summation extending over all the roots α of the equation $\Delta(p) = 0$, the accent denoting differentiation with respect to p.

$$q e^{-qx} = \frac{1}{\sqrt{\pi k t}} e^{-x^2/4kt} \qquad (x > 0) \quad \ldots \quad (3)$$

2. To shew the advantage of using the operational method, we may apply it to verify the expression for the temperature in an infinite rod in which the conductivity is K_1 for $x > 0$ and K_2 for $x < 0$, and there is an instantaneous source of strength unity at $x = \xi$ (> 0).

Let k_1, k_2 be the diffusivities in the two portions and put

$$p = k_1 q_1^2 = k_2 q_2^2.$$



If V_1, V_2, V_3 be the temperatures in the portions $0 < x < \xi$, $x > \xi$, and $x < 0$; we assume

$$V_1 = A e^{q_1 x} + B e^{-q_1 x};$$

$V_2 = C e^{-q_1 x}$ since this should vanish when $x \to \infty$;

$V_3 = D e^{q_2 x}$ since this should vanish when $x \to -\infty$.

The conditions at the junction $x = 0$ are:

$$V_3 = V_1 \quad \text{and} \quad K_2 \frac{\partial V_3}{\partial x} = K_1 \frac{\partial V_1}{\partial x}$$

which give
$$D = A + B$$

and
$$K_2 q_2 D = K_1 q_1 (A - B);$$

or
$$A - B = D \cdot \frac{K_2 q_2}{K_1 q_1} = D \frac{K_2}{K_1} \sqrt{\frac{k_1}{k_2}} = \frac{D}{\alpha} \text{ (say).}$$

Again at the source $x = \xi$, we should have

$$V_1 = V_2 \text{ and } \frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x} = q_1^2$$

which give
$$A e^{q_1\xi} + B e^{-q_1\xi} = C e^{-q_1\xi}$$

$$A e^{q_1\xi} - B e^{-q_1\xi} = q_1 - C e^{-q_1\xi}$$

Solving these four equations for A, B, C and D, we get

$$A = \tfrac{1}{2} q_1 e^{-q_1\xi}; \qquad B = \tfrac{1}{2}\frac{a-1}{a+1} q_1 e^{-q_1\xi}$$

$$C = \tfrac{1}{2} q_1 e^{q_1\xi} + \tfrac{1}{2}\frac{a-1}{a+1} q_1 e^{-q_1\xi}$$

$$D = \frac{a}{a+1} q_1 e^{-q_1\xi}$$

Thus
$$V_1 = \tfrac{1}{2} q_1 e^{-q_1(\xi - x)} + \tfrac{1}{2}\frac{a-1}{a+1} q_1 e^{-q_1(\xi + x)}$$

$$V_2 = \tfrac{1}{2} q_1 e^{-q_1(x-\xi)} + \tfrac{1}{2}\frac{a-1}{a+1} q_1 e^{-q_1(x+\xi)}$$

$$V_3 = \frac{a}{a+1} q_1 e^{-q_1\xi + q_2 x}$$

$$= \frac{a}{a+1}\sqrt{\frac{k_2}{k_1}}\, q_2 e^{-q_2\left(\xi\sqrt{\frac{k_2}{k^1}} - x\right)}.$$

Interpreting these by the rule (3) above, we have for V_1 and V_2,

$$V_{1,2} = \frac{1}{2\sqrt{\pi k_1 t}}\left[e^{-\frac{(x-\xi)^2}{4k_1 t}} + \frac{a-1}{a+1}\cdot e^{-\frac{(x+\xi)^2}{4k_1 t}}\right] \quad \ldots \quad (4)$$

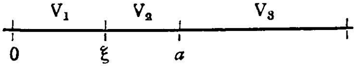
and for V_3,

$$V_3 = \frac{a}{a+1}\frac{1}{\sqrt{\pi k_1 t}} e^{-\frac{(x-\xi\sqrt{k_2/k_1})^2}{4k_2 t}} \quad \ldots \quad (5)$$

which are the known expressions for the temperatures in the two portions, usually obtained after laborious analysis (see Carslaw, pp. 161—163).

3. The case of a finite rod is treated similarly. Let the length of the rod be $(a + b)$ and let K, k be the conductivity and diffusivity from $x = 0$ to $x = a$, and K_1, k_1 those from $x = a$ to $x = b$. Let the ends

of the rod be maintained at zero temperature while there is an instantaneous source of strength unity at $x = \xi\ (< a)$.



If V_1, V_2, V_3 be the temperatures from 0 to ξ, ξ to a, and a to b respectively, V_1 and V_2 satisfy the equation

$$\frac{\partial^2 V}{\partial x^2} = q^2 V$$

and V_3 satisfies :

$$\frac{\partial^2 V}{\partial x^2} = q_1{}^2 V$$

where
$$\frac{\partial}{\partial t} = p = kq^2 = k_1 q_1{}^2.$$

Hence we assume

$$V_1 = \Big[A \cosh q\,(a - \xi) + B \sinh q\,(a - \xi)\Big] \frac{\sinh q x}{\sinh q \xi},$$

$$V_2 = A \cosh q\,(a - x) + B \sinh q\,(a - x),$$

$$V_3 = A\,\frac{\sinh q_1\,(a + b - x)}{\sinh q_1\, b},$$

since
$$V_1 = 0 \text{ at } x = 0$$

$$V_1 = V_2 \text{ at } x = \xi$$

$$V_2 = V_3 \text{ at } x = a$$

and
$$V_3 = 0 \text{ at } x = a + b.$$

The condition at the junction of the two materials is :

$$K\,\frac{\partial V_2}{\partial x} = K_1\,\frac{\partial V_3}{\partial x} \quad \text{at } x = a,$$

and the condition at the source is

$$k\left(\frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x}\right) = p = kq^2 \quad \text{at } x = \xi$$

or
$$\frac{\partial V_1}{\partial x} - \frac{\partial V_2}{\partial x} = q^2 \quad \text{at } x = \xi.$$

These give

$$K\, q\, B \sinh q_1 b = K_1\, q_1\, A \cosh q_1 b$$

or if we put

$$\alpha = \frac{K_1 q_1}{K q} = \frac{K_1}{K}\sqrt{\frac{k}{k_1}},$$

$$A\, \alpha \cosh q_1 b - B \sinh q_1 b = 0$$

and

$$A \cosh qa + B \sinh qa = q \sinh q\xi.$$

From the last two equations we have

$$\frac{A}{q \sinh q\xi \sinh q_1 b} = \frac{B}{\alpha q \sinh q\xi \cosh q_1 b}$$

$$= \frac{1}{\cosh qa \sinh q_1 b + \alpha \sinh qa \cosh q_1 b};$$

whence

$$V_1 = \frac{q \sinh qx}{\Delta}\left[\sinh q_1 b \cosh q(a-\xi) + \alpha \cosh q_1 b \sinh q(a-\xi)\right],$$

$$V_2 = \frac{q \sinh q\xi}{\Delta}\left[\sinh q_1 b \cosh q(a-x) + \alpha \cosh q_1 b \sinh q(a-x)\right],$$

$$V_3 = \frac{q \sinh q\xi}{\Delta} \sinh q_1 (a + b - x),$$

where for shortness we have written

$$\Delta = \cosh qa \sinh q_1 b + \alpha \sinh qa \cosh q_1 b$$

since $k_1 q_1^2 = k q^2,\ q_1 = \sqrt{\frac{k}{k_1}}\, q = \sigma q$ if $\sigma = \sqrt{k/k_1}$

and $\Delta = \cosh qa \sinh \sigma q b + \alpha \sinh qa \cosh \sigma q b.$

These operational solutions have to be interpreted by Heaviside's formula (2).

The roots of $\Delta = 0$ are infinite in number, purely imaginary, and not repeated; and to every positive root corresponds an equal negative root.

The roots are therefore given by

$$q = \lambda i,\ p = kq^2 = -k\lambda^2$$

where λ is a root of the equation

$$\alpha \tan a\lambda + \tan \sigma b \lambda = 0 \quad \ldots \quad \ldots \quad (6)$$

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NOTES AND QUESTIONS.

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Notes and Questions.

On the conditions for the Double Contact of Two Conics.*

1. Let us first note down the conditions for

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

representing two coincident straight lines.

Identifying S with $(lx + my + nz)^2$, we have

$$a : b : c : f : g : h = l^2 : m^2 : n^2 : mn : nl : lm.$$

Hence $bc - f^2$, $ca - g^2$, $ab - h^2$, $gh - af$, $hf - bg$, $fg - ch$ are all equal to zero. These are therefore *necessary* conditions, for S being a "square." They are *sufficient* conditions as well. For, if they hold, we have

$$S \equiv \frac{1}{a}(ax + hy + gz)^2, \text{ when } a \neq 0;$$

$$\equiv \frac{1}{b}(hx + by + fz)^2, \text{ when } b \neq 0; \text{ and}$$

$$\equiv \frac{1}{c}(gx + fy + cz)^2, \text{ when } c \neq 0.$$

There are, indeed, other forms of writing S as a square. Thus, when all the coefficients occur, we could write

$$S \equiv fgh\left(\frac{x}{f} + \frac{y}{g} + \frac{z}{h}\right)^2;$$

in which, without loss of symmetry, we could have written abc for the multiplier fgh.

2. Let $S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$

and $S' \equiv a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy$

be two conics which, in the first instance, we shall consider to be proper conics, that is, having discriminants Δ, $\Delta' \neq 0$. They will have

* See Salmon's *Conic Sections*, Art. 378*a*.

double contact with one another if $kS + S'$ will represent two coincident straight lines for some value of k. This value, obviously, cannot be 0 (or ∞). It must therefore be a finite value other than 0. The necessary and sufficient conditions for $kS + S'$ representing two coincident straight lines are, by § 1,

$$\left.\begin{aligned} &(kb+b')(kc+c')-(kf+f')^2=0;\\ &\qquad \text{and two similar equations;}\\ \text{and}\quad &(kg+g')(kh+h')-(ka+a')(kf+f')=0;\\ &\qquad \text{and two similar equations.}\end{aligned}\right\} \quad \ldots \quad (2.1)$$

Written out, these are

$$\left.\begin{aligned} &k^2(bc-f^2)+k(bc'+b'c-2ff')+(b'c'-f'^2)=0\\ &k^2(ca-g^2)+k(ca'+c'a-2gg')+(c'a'-g'^2)=0\\ &k^2(ab-h^2)+k(ab'+a'b-2hh')+(a'b'-h'^2)=0\\ &k^2(gh-af)+k(gh'+g'h-af'-a'f)+(g'h'-a'f')=0\\ &k^2(hf-bg)+k(hf'+h'f-bg'-b'g)+(h'f'-b'g')=0\\ &k^2(fg-ch)+k(fg'+f'g-ch'-c'h)+(f'g'-c'h')=0\end{aligned}\right\} \quad (2.2)$$

which may be regarded as the fundamental equations for the double contact of S and S'.

In these equations the co-efficients of k^2 are the quantities usually denoted by A, B, C, F, G, H; and the absolute terms are denoted by A', B', C', F', G', H'. Let us denote the coefficients of k by $a_0, b_0, c_0, f_0, g_0, h_0$. These are the coefficients of the conic Φ.

Eliminating k^2 and k linearly from these equations, we see that all the determinants of the third order in the following matrix are zero; or briefly, the matrix itself is zero.

That is, we have

$$\left\| \begin{matrix} A, & B, & C, & F, & G, & H \\ a_0 & b_0 & c_0 & f_0 & g_0 & h_0 \\ A' & B' & C' & F' & G' & H' \end{matrix} \right\| = 0 \ldots \quad (2.3)$$

The vanishing of the matrix (2.3) thus constitutes a set of necessary conditions for the double contact of S and S′. Later on we shall show that they are sufficient conditions as well.

3. Multiplying the equations in (2.2) by a, b, c, $2f$, $2g$, $2h$ respectively, and adding, we get

$$k^2 . 3\Delta + 2k . \Theta + \Theta' = 0 \qquad \dots \quad (3.1)$$

Multiplying them by a', b', c', $2f'$, $2g'$, $2h'$ and adding we get

$$k^2 . \Theta + 2k . \Theta' + 3\Delta' = 0. \qquad \dots \quad (3.2)$$

Adding this to k times (3.1), and removing the factor 3, we get

$$\Delta k^3 + \Theta k^2 + \Theta' k + \Delta' = 0. \qquad \dots \quad (3.3)$$

This last is Lamè's equation for $kS + S'$; and it shows that our k is one of the roots of Lamè's equation. The equation (3.1) is simply the derivate of (3.3) with respect to k. Hence, k is a double root of Lamè's equation.

From (3.1) and (3.2), by eliminating k, we see that the tact-invariant of S and S′ vanishes; showing that S touches S′. These are verifications of known facts from our fundamental equations (2.2).

Let λ, μ, ν be any three quantities. Multiplying the equations in (2.2) by λ^2, μ^2, ν^2, $2\mu\nu$, $2\nu\lambda$, $2\lambda\mu$ and adding, and using known notations we get

$$k^2 . \Sigma + k\Phi + \Sigma' = 0. \qquad \dots \quad (3.4)$$

Eliminating k^2 and k linearly between (3.4), (3.1) and (3.2) we get

$$\begin{vmatrix} \Sigma & \Phi & \Sigma' \\ 3\Delta & 2\Theta & \Theta' \\ \Theta & 2\Theta' & 3\Delta' \end{vmatrix} = 0 \qquad \dots \quad (3.5)$$

We thus obtain the identical relation holding between Σ, Φ and Σ' when there is double contact between S and S′. Finally, eliminating Θ' between (3.1) and, say, (3.3) we get

$$3k^3 . \Delta + 2k^2\Theta = -\Theta' k = \Delta k^3 + \Theta k^2 + \Delta',$$

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Hence

$$\Delta' = 2\,\Delta k^3 + \Theta k^2, \qquad \dots \quad (3.6)$$

a relation that we shall make use of hereafter.

4. We now proceed to an important transformation of the equations (2.2) and obtain from them the following, namely:

$$\left.\begin{array}{l} k^2 . \Delta a' + k . (BC' + B'C - 2\,FF') \qquad + \Delta' a = 0 \\ k^2 . \Delta b' + k . (CA' + C'A - 2\,GG') \qquad + \Delta' b = 0 \\ k^2 . \Delta c' + k . (AB' + A'B - 2\,HH') \qquad + \Delta' c = 0 \\ k^2 . \Delta f' + k . (GH' + G'H - AF' - A'F) + \Delta' f = 0 \\ k^2 . \Delta g' + k\ (HF' + H'F - BG' - B'G) + \Delta' g = 0 \\ k^2\ \ \Delta h' + k . (FG' + F'G - CH' - C'H) + \Delta' h = 0 \end{array}\right\} \qquad \dots \quad (4.1)$$

To prove the first of these results, we take the 2nd, 3rd and 4th of the equations in (2.2), namely

$$k^2 . B + k\,(ca' + c'a - 2\,gg') \qquad + B' = 0$$

$$k^2 . C + k\,(ab' + a'b - 2\,hh') \qquad + C' = 0$$

$$k^2 . F + k\,(gh' + g'h - af' - a'f) + F' = 0$$

Multiply the first of these by C, the second by B, the third by —2F, and add; then multiply by k. Remembering that $BC - F^2 = \Delta . a$, we get

$$2k^3 . \Delta a + k^2 . [\quad] + (BC' + B'C - 2\,FF')k = 0 \qquad \dots \quad (4.2)$$

The coefficient of k^2 is

$$C\,(ca' + c'a - 2\,gg') + B\,(ab' + a'b - 2\,hh')$$

$$- 2F\,(gh' + g'h - af' - a'f).$$

Rearrange this as a linear function of a', b', c', f', g', h', and we find the coefficient of k^2 to be

$$a'(\Delta + Aa) + b' . Ba + c' . Ca + 2f' . Fa + 2g' . Ga + 2h' . Ha,$$

that is, it is equal to $a'\,\Delta + a\,\Theta$.

Consequently (4.2) becomes

$$2k^3 . \Delta a + k^2 (a'\Delta + a\Theta) + (BC' + B'C - 2FF') k = 0 \ldots \quad (4.3)$$

But, by (3.6), $\Delta' = 2\Delta k^3 + \Theta k^2.$

Therefore we have

$$k^2 \Delta a' + (BC' + B'C - 2FF') k + \Delta' a = 0$$

which was to be proved.

To prove the 4th equation in (4.1) we take the 1st, 4th, 5th and 6th equations in (2.2), namely,

$$k^2 . A + k (bc' + b'c - 2ff') \qquad + A' = 0$$

$$k^2 . F + k (gh' + g'h - af' - a'f) + F' = 0$$

$$k^2 . G + k (hf' + h'f - bg' - b'g) \quad + G' = 0$$

$$k^2 . H + k (fg' + f'g - ch' - c'h) \quad + H' = 0$$

We multiply the first of these by $-$ F, the second by $-$ A, the third by H, and the fourth by G; and add; and multiply by k. Then reducing in the above manner we get the fourth equation in (4.1), the result in (3.6) coming in again for application. The equations (4.1) are thus proved.

It may be observed that the equations (4.1) are equivalent to our fundamental equations (2.2). The one set can be deduced from the other when Δ, Δ' are not equal to zero.

5. Eliminating $k^2 \Delta$, k and Δ' linearly from the equations (4.1), and using $A_0, B_0, C_0, F_0, G_0, H_0$ to denote the coefficients of k therein, (they are the coefficients of the **F**-conic), we get

$$\begin{vmatrix} a, & b, & c, & f, & g & h \\ A_0 & B_0 & C_0 & F_0 & G_0 & H_0 \\ a' & b' & c' & f' & g' & h' \end{vmatrix} = 0 \quad \ldots \quad (5.1)$$

agreeing with the result in Salmon's *Conic Sections*, Art. 378*a*.

Proceeding after the manner in § 3, we can, from the equations (4.1), deduce that the following identical relation holds between the conics S, S′ and **F** when there is double contact between S and S′, namely

$$\begin{vmatrix} S' & \mathbf{F} & S \\ 3\Delta' & 2\Delta'\Theta & \Theta' \\ \Theta & 2\Delta\Theta' & 3\Delta \end{vmatrix} = 0 \qquad \ldots \quad (5.2)$$

6. We shall suppose now that our conic S′ consists of two *intersecting* straight lines. This will make $\Delta' = 0$. But it can be seen that our derivation of the equations (4.1) remains unaffected. Putting $\Delta' = 0$ therein, we see that when a conic S has double contact with the conic S′ which breaks into two intersecting lines, we have

$$\frac{BC' + B'C - 2FF'}{a'} = \frac{CA' + C'A - 2GG'}{b'} = \frac{AB' + A'B - 2HH'}{c'}$$

$$= \frac{GH' + G'H - AF' - A'F}{f'} = \frac{HF' + H'F - BG' - B'G}{g'}$$

$$= \frac{FG' + F'G - CH' - C'H}{h'}. \qquad \ldots \quad (6.1)$$

This means that the conic **F** becomes identical with S′ if S′ denote a pair of intersecting lines touching S. This can be otherwise seen as follows. When S′ breaks up into two intersecting lines (intersecting in O, say), then **F**, from its geometrical property, breaks up into the pair of tangents drawn to S from O. Hence if the two lines constituting S′ themselves touch S, then **F** becomes identical with S′.

7. We shall now give some attention to the case when S and S′ both break into pairs of intersecting lines. Let O, O′ be the points of intersection of the lines constituting S, S′. In this case it can be seen that no conic of the system $k\,S + S'$ can become a "square", that is, represent two coincident lines so long as O and O′ are distinct. That is, the condition for double contact requires that O and O′ are identical. Further, when O and O′ are identical there are *two values* of k for which $k\,S + S'$ will become a "square".

In the equations (4.1) putting $\Delta = 0$ and $\Delta' = 0$ we see that the lines represented by S will have the same point of intersection as the lines represented by S′ *provided the co-efficients* $A_0, B_0, C_0, F_0, G_0, H_0$ *of the* **F** *conic are all zero.*

Again as there are two values of k in this case for which $k\,S + S'$ becomes a "square"; and as each of these values should satisfy each of our fundamental equations (2.2); we see that every one of these equations has the same two roots as any of the others. Hence we have the theorem that if S, S′ represent intersecting lines having a common point of intersection O, then, the ratios A : B : C : F : G : H are equal to the ratios A′ : B′ : C′ : F′ : G′ : H′ as well as to the ratios

$$a_0 : b_0 : c_0 : f_0 : g_0 : h_0$$

these last quantities being the co-efficients of the Φ conic of S, S′.

In the same case, if the lines constituting S separate harmonically the lines constituting S′, then the square members of the system $k\,S + S'$ will be of the forms $k_1\,S + S'$ and $-k_1\,S + S'$. Hence each of the quadratics in (2.1) has the sum of its roots equal to zero. Therefore in the equations in (2.1) the co-efficients of k would be all zero Hence we have the theorem that if S, S′ break up into four concurrent lines and those forming S separate those forming S′ harmonically, then, the co-efficients $a_0, b_0, c_0, f_0, g_0, h_0$ of the Φ conic of S, S′ are all equal to zero.

It may be noted that, in this case, every line in the plane is cut by S, S′ harmonically ; and that is why Φ becomes a null-conic.

The Hyperosculation of S and S′.

8. Let us revert to the case of two *proper* conics S, S′ having double contact with one another. Let the value k make $kS + S'$ a "square." Then, supposing $ka + a'$ is not zero, the chord of contact, by § 1, will have for its equation

$$(ka + a')x + (kh + h')y + (kg + g')z = 0. \quad \dots \quad (8.1)$$

If this chord of contact be a tangent to S it will be a case of hyperosculation of S and S′. The tangential equation of S being

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0$$

the line (8.1) will touch S provided,

$$A(ka + a')^2 + B(kh + h')^2 + C(kg + g'^2)$$

$$+ 2F(kh+h')(kg+g') + 2G(kg+g')(ka+a') + 2H(ka+a')(kh+h') = 0. \qquad \ldots (8.2)$$

Dividing each term by $ka+a'$ (which is not zero), and using the equations (2.1) we get

$$A(ka + a') + B(kb + b') + C(kc + c')$$

$$+ 2F(kf + f') + 2G(kg + g') + 2H(kh + h') = 0. \qquad \ldots (8.3)$$

that is, we have

$$k.3\Delta + \Theta = 0. \qquad \ldots (8.4)$$

Recalling the equations (3.1) and (3.2) that k satisfies in virtue of the double contact, namely,

$$k^2.3\Delta + 2k.\Theta + \Theta' = 0,$$

$$k^2.\Theta + 2k.\Theta' + 3\Delta' = 0,$$

and depressing these by use of (8.4) we get

$$k\Theta + \Theta' = 0 \qquad \ldots (8.5)$$

$$k\Theta' + 3\Delta' = 0 \qquad \ldots (8.6)$$

From these we see that k is a triple root of Lamè's equation.

The equations (3.1) and (3.2) having been deduced from (2.2), we see that the equations (8.4), (8.5) and (8.6) are equivalent to one another. Any one of these, coupled with the equations (2.2), constitute the conditions for hyperosculation.

Summary.

9. The discussion above enables us to distinguish the cases of various kinds of contact that may exist between two proper conics S, S′.

The discriminant of $kS + S'$ is

$$\begin{vmatrix} ka + a', & kh + h', & kg + g' \\ kh + h', & kb + b', & kf + f' \\ kg + g', & kf + f', & kc + c' \end{vmatrix}$$

whose vanishing constitutes Lamè's equation.

(i) If all the first minors in the discriminant have a common root k, then S and S′ have double contact with one another. For, their vanishing for a common value k gives us the fundamental equations (2.2) which are the necessary and sufficient conditions for the double contact of S and S′.

(ii) In that case k is necessarily a double root of Lamè's equation, as shown in para 3.

(iii) If k is a double root of Lamè's equation, without being a root of all the first minors also, then S and S′ have single contact with one another.

(iv) If all the first minors in the discriminant have a common root k, and k is a triple-root of Lamè's equation, then S and S′ hyperosculate one another, by para 8.

(v) If k is a triple-root of Lamè's equation, without being a root of all the first minors also, then S and S′ osculate one another.

In this connection, the reader may see Bromwich's *Quadratic Forms and their classification by means of Invariant Factors*—[Cambridge Tracts in Mathematics and Mathematical Physics, No. 3.]

10. I shall defer to a further note a proof of the statement that the vanishing of the determinants of the matrix (2.3) is sufficient to secure the double contact of S and S′.

V. RAMASWAMI AIYAR.

An Application of the Permanent.

1. The *permanent* of a square matrix $|a_{rs}|$ $(r, s = 1, 2, \ldots n)$ is defined to be the function, which has the *same* terms as the determinant, but with all the signs positive. While one can hardly take a step in any branch of mathematics, without knocking against the determinant, the applications of the permanent are few.

The present note shows that the number of solutions of a system of equations in several sets of homogeneous variables, an be expressed in terms of the permanent formed from the orders of the equations in the several variables. I have not met with the result before, and believe it to be new.

2. A curve on a quadric surface is characterised by two partial orders (m, n), which represent the number of its intersections with the generators of two systems on the surface. It is known that two curves on the same quadric, of partial orders (m, n), (m', n') intersect in $mn' + m'n$ points; this is equivalent to saying that the number of common solutions of two double-binary equations of partial orders (m, n), (m', n') is equal to the permanent

$$\begin{vmatrix} m & n \\ m' & n' \end{vmatrix}.$$

Considering more general systems of equations of the same type, let $X^{(p)}$ represent a p-ary variable, that is, an assemblage of p homogeneous variables $X_1^{(p)}, X_2^{(p)}, \ldots, X_p^{(p)}$. Consider a space S, the points of which can be represented uniquely, without repetition or exception, by t independent variables

$$X^{(p_1)}, X^{(p_2)}, \ldots, X^{(p_t)}.$$

Such a space is of

$$p_1 - 1 + p_2 - 1 + \ldots + p_t - 1 = p_1 + p_2 + \ldots + p_t - t$$

dimensions; it is the natural extension of a quadric surface, and may be called *the multi-linear space* of dimensions

$$(p_1 - 1, p_2 - 1, \ldots, p_t - 1).$$

An algebraic locus Γ in S will be represented by an equation, which is homogeneous in each of the sets of variables

$$X^{(p_1)}, X^{(p_2)}, \ldots X^{(p_t)},$$

and which is (say) of order l_k in the variables $X^{(p_k)}$ $(k = 1, 2, \ldots, t)$.

We shall say that Γ has the partial orders $l_1, l_2 \dots l_t$. Since S is of $p_1 + p_2 + \dots + p_t - t$ dimensions, it follows that $p_1 + p_2 + \dots + p_t - t$ loci like Γ will have a finite number of intersections. It is the question to find this number as a function of the partial orders.

3. If $t = 1$, S becomes a *uni-linear space*, that is, an ordinary projective space of $p_1 - 1$ dimensions; in this case, algebraic loci in S have only one order, and the number of intersections of $p_1 - 1$ loci is simply the product of their orders.

The form of the general formula is made apparent by first considering the case when all the variables are binary ($p_1 = p_2 = \dots = p_t = 2$): S is then, what we may call *multi-binary space* of t dimensions, being the immediate extension of the quadric surface. Let us denote by the symbol

$$\begin{bmatrix} l_1 & l_2 & \dots & l_t \\ m_1 & m_2 & \dots & m_t \\ \dots & \dots & \dots & \dots \\ x_1 & x_2 & \dots & x_t \end{bmatrix}.$$

the number of intersections of t algebraic loci in S, whose partial orders are $(l_1, l_2, \dots, l_t)$, $(m_1, m_2, \dots, m_t)$, $\dots$ $(x_1, x_2, \dots, x_t)$. From the number of intersections of two curves on a quadric, it follows that the symbol $\begin{bmatrix} \lambda & \mu \\ \lambda' & \mu' \end{bmatrix}$ has the value $\lambda\mu' + \lambda'\mu$. To evaluate the symbol in the general case, take (X_1, X_2), (Y_1, Y_2), $\dots$ as the t binary variables, and suppose that the first of the t algebraic loci takes the form

$$X_1^{l_1} Y_1^{l_2} \dots = 0.$$

Since the number of intersections depends only on the partial orders, it will not be affected by this supposition. The solution of the t equations then leads to the development:

$$\begin{bmatrix} l_1 & l_2 & \dots & l_t \\ m_1 & m_2 & \dots & m_t \\ \dots & & & \\ x_1 & x_2 & \dots & x \end{bmatrix} = l_1 \begin{bmatrix} m_2 & \dots & m_t \\ \dots & & \\ \dots & & \\ x_2 & \dots & x_t \end{bmatrix} + l_2 \begin{bmatrix} m_1 & m_3 & \dots & m_t \\ \dots & & & \\ \dots & & & \\ x_1 & x_3 & \dots & x_t \end{bmatrix} + \text{etc.}$$

Since $\begin{bmatrix} \lambda \mu \\ \lambda' \mu' \end{bmatrix}$ has the value $\lambda \mu' + \lambda' \mu$, it follows at once by induction that the symbol on the left represents the *permanent* function of its elements.

4. When we examine the general case, we find that an analogous result holds. The result stated in algebraic form is :

The number of solutions of $p_1 + p_2 + \ldots + p_t - t$ equations in t variables $X^{(p_1)}, X^{(p_2)}, \ldots \ldots, X^{(p_t)}$ where $X^{(p_k)}$ is a p_k-ary variable is equal to

$$\frac{1}{(p_1 - 1)!\,(p_2 - 1)!\ldots(p_t - 1)!}\cdot P,$$

where P is the permanent formed from the partial orders of the equations, the partial order in respect of a m-ary variable being written $m-1$ times in succession ($m = p_1, p_2, \ldots, p_t$)

The theorem may be proved inductively by the enumerative device, already adopted, of choosing a special form for one of the equations. Let the p_k variables $X^{(p_k)}$ be denoted by

$$X_1^{p_k},\ X_2^{p_k},\ \ldots\ X_{p_k}^{p_k};$$

and let the partial orders of the equations be respectively

$$(l_1, l_2, \ldots, l_t), (m_1, m_2, \ldots, m_t) \ldots, (x_1, x_2, \ldots, x_t).$$

Let us denote the required number of solutions by the symbol N.

Let the first of our equations be chosen to have the special form:

$$(X_1^{p_1})^{l_1} (X_1^{p_2})^{l_2} \ldots (X_1^{p_t})^{l_t} = 0$$

It is clear then that the required number of solutions, N, is given by

$$N = l_1 N_1 + l_2 N_2 + \ldots + l_t N_t \qquad \ldots \quad (1)$$

where N_k is the number of common solutions of the remaining equations of the system, and the equation $X_1^{p_k} = 0$. The result of putting $X_1^{p_k} = 0$ in the remaining equations is simply to replace the p_k-ary variable $X^{(p_k)}$ by a $(p_k - 1)$-ary variable.

Now from the previous paragraph, the theorem is obviously true for the case of t binary variables. We shall assume the theorem to be true for values of $p_1\, p_2 \ldots p_t$ such that $p_1+p_2+\ldots+p_t-2t$ is equal to a given number m. Taking values of $p_1, p_2, \ldots, p_t$ such that $p_1+p_2+\ldots+p_t-2t = m+1$, the numbers N_k in (1) can be evaluated, on account of our assumption, by means of the theorem; namely:

$$N_1 = \frac{1}{(p_1-2)!\,(p_2-1)!\,\ldots\,(p_t-1)!} \times \text{permanent} \begin{vmatrix} \overset{(p_1-2)}{m_1} & \overset{(p_2-1)}{m_2} & \ldots & \overset{(p_t-1)}{m_t} \\ \ldots & \ldots & \ldots & \ldots \\ x_1 & x_2 & \ldots & x_t \end{vmatrix}$$

N_k = Similar expression,

(the numbers (p_1-2), $(p_2-1)\ldots$ on the top indicating the number of times the corresponding column is to be written).

Hence

$$N = l_1 N_1 + l_2 N_2 + \ldots + l_t N_t$$

$$= \frac{1}{(p_1-1)!\,(p_2-1)!\ldots(p_t-1)!} \times$$

$$\{ (p_1-1)\, l_1 N_1 + (p_2-1)\, l_2 N_2 + \ldots + (p_t-1) l_t N_t \}.$$

But the expression within the brackets is the expansion of the permanent

$$\begin{vmatrix} \overset{(p_1-1)}{l_1} & \overset{(p_2-1)}{l_2} & \ldots & \overset{(p_t-1)}{l_t} \\ m_1 & m_2 & \ldots & m_t \\ \ldots & \ldots & \ldots & \ldots \\ x_1 & x_2 & \ldots & x_t \end{vmatrix}$$

Thus the theorem is proved true for values of $p_1, p_2, \ldots, p_t$ such that $p_1+p_2+\ldots+p_t-2t = m+1$. Since the theorem is true for the case of t binary variables, the induction is complete and it is true universally.

In particular, if all the equations are multi-linear, the number of solutions is the multinomial co-efficient

$$\frac{(p_1 + p_2 + \dots + p_t - t)!}{(p_1 - 1)!\,(p_2 - 1)!\dots(p^t - 1)!}.$$

This is because the value of the permanent of order n, all of whose elements are 1, is equal to the number of terms in a determinant of the same order, that is, to n!

5. A word may be added about the multi linear spaces, which we have used as our geometrical background. A multi-linear space may always be conceived as an algebraic manifold in a projective space of higher dimensions. Thus the double-binary space may be conceived as a quadric in three space, the co-ordinates of any point on the quadric being bi-linear forms in two binary variables. More generally, the number of linearly independent multi-linear forms in the t variables $X^{(p_1)}, \dots, X^{(p_t)}$, being clearly $p_1\, p_2 \dots p_t$, the multilinear space of dimensions

$$(p_1 - 1,\ p_2 - 1,\ \dots\ p_t - 1)$$

can be conceived as an algebraic manifold Γ of

$$(p_1 + p_2 + \dots + p_t - t)$$

dimensions imbedded in a flat space of $(p_1\, p_2 \dots p_t - 1)$ dimensions. The co-ordinates of any point on Γ are multilinear forms in the t variables. It obviously follows that Γ possesses t distinct systems of generating regions, those of the kth system being of $(p_k - 1)$ dimensions; one and only one generating region of each system passes through each point of Γ. Distinct generating regions of the same system do not intersect, while those of different systems may intersect.

The order of Γ is clearly the multinomial co-efficient already obtained as the number of solutions of $(p_1 + p_2 + \dots + p_t - t)$ multi-linear equations in $X^{(p_1)}, \dots, X^{(p_t)}$.

For the case $t = 2$, the loci Γ have been investigated by various authors For properties and references, the paper of Richmond and Bath : 'On loci which have two systems of generating spaces' may be consulted.—(*Proc. Camb. Phil. Soc.* p. 319. Vol 22. Sep. 1924.)

R. VAIDYANATHASWAMY.

Some Identities in the Theory of Numbers.

1. Let $f(x)$ and $F(x)$ be defined as follows:

(i) $f(mn) = f(m)f(n)$, for any two positive integers m, n.

(ii) If n is a positive integer

$$F(n) = \sum_{d/n} f(d)$$

(iii) $f(x) = F(x) = 0$ if x is not an integer.

The object of this note is to prove the following result:

$$F(uv) = \sum_{1}^{\infty} \mu(n)\, f(n)\, F\left(\frac{u}{n}\right) F\left(\frac{v}{n}\right)$$

where u, v are any two positive integers, $\mu(n)$ is Merten's function.

This result was found independently by Dr. Vaidyanathaswamy and myself.

If we put $f(n) = n^a$ here, we get

$$\sigma_a(uv) = \sum_{n=1}^{\infty} \mu(n)\, n^a\, \sigma_a\left(\frac{u}{n}\right) \sigma_a\left(\frac{v}{n}\right)$$

where $\sigma_a(x)$ is the sum of the ath powers of the divisors of x, when x is a positive integer, and is zero otherwise.

The case when a is zero was found by Ramanujan.*

2. Let

$$L(s) = \sum_{1}^{\infty} \frac{f(n)}{n^s} = \Pi \left\{1 - f(p)\, p^{-s}\right\}^{-1}$$

then

$$\sum_{1}^{\infty} \frac{F(n)}{n^s} = \zeta(s)\, L(s) = \Pi \left\{ \sum_{m=0}^{\infty} S_m(p)\, p^{-ms} \right\}$$

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* See *Mess. of Maths.*, 45, (1916), 81–84; and *Proc. Lond. Math. Soc.* (1923) 235–251.

where p runs through all primes, $\zeta(s)$ is Riemann's Zeta Function, and $S_m(p)$ is equal to

$$\sum_{k=0}^{k=m} \{f(p)\}^k$$

Hence

$$\sum_1^\infty \frac{F(nv)}{n^s} = \prod_\theta \left\{ \sum_{m=0}^\infty S_m(\theta)\,\theta^{-ms} \right\} . \prod_\phi \left\{ \sum_{m=0}^\infty S_{m+\beta}(\phi)\,\phi^{-ms} \right\}$$

where v is a positive integer equal to $\Pi(\phi^\beta)$ where the ϕ's are distinct primes, and θ runs through all the primes which are not factors of v. By a little easy algebra we see that the second member of this expression is equal to

$$\prod_\theta \left\{ \sum_0^\infty S_m(\theta)\,\theta^{-ms} \right\} \prod_\phi \left\{ \sum_0^\infty S_m(\phi)\,\phi^{-ms} \right\}$$

$$\times \prod_\phi \left\{ S_\beta(\phi) - f(\phi)\,S_{\beta-1}(\phi)\,\phi^{-s} \right\}$$

Since the product of the first two factors is

$$\sum_1^\infty \frac{F(n)}{n^s}$$

the product of the three factors is

$$\sum_1^\infty \frac{F(n)}{n^s} \times \sum_{\delta=1}^\infty \frac{\mu(\delta)\,f(\delta)}{\delta^s}\,F\left(\frac{v}{\delta}\right)$$

Hence

$$\sum_1^\infty \frac{F(nv)}{n^s} = \left(\sum_1^\infty \frac{F(n)}{n^s}\right) \times \left(\sum_1^\infty \frac{\mu(\delta)\,f(\delta)}{\delta^s}\,F\left(\frac{v}{\delta}\right)\right)$$

Equate the co-efficients of n^{-s} on both sides. Hence

$$F(uv) = \sum_{\delta=1}^\infty \mu(\delta)\,f(\delta)\,F\left(\frac{u}{\delta}\right)F\left(\frac{v}{\delta}\right)$$

Our theorem is proved.

S. D. CHOWLA.

A Prize Problem.

The attention of our readers is invited to Question No. 1547 appearing on p. 95 of this issue. It has been proposed by our President, Mr. V. Ramaswami Aiyar, who hopes to see a solution of it simpler than his own, and offers a medal for the best investigation of it, received before the 1st January 1930.

Members as well as non-members may compete. Solutions should be sent to the Editor, in an envelope marked "Prize Problem." They will be scrutinized by a Committee whose decision will be final.

EDITOR.

Review.

The Theory of Equations and the Complex Variable, by RAMCHARAN BISWAS. (*Chuckervarthy Chatterjee & Co., Calcutta, 1926*).

The scope of the book is substantially the same as that of Burnside and Panton: *Theory of Equations*, Vol. 1. In respect of treatment, this book differs from Burnside and Panton's, in being somewhat less elaborate, and confining itself to a selected material; for this reason it will probably be of help to students preparing for degree examinations.

A novel feature of the book is 'the biradical transformation of an equation' by means of which, the equation giving the sums of two roots of a given equation and the equation of squared differences are obtained simultaneously The importance and usefulness of this transformation are amply demonstrated in the book.

As the book professes to be elementary, no attempt is made to introduce the notion of 'group,' or to furnish methods of deriving from the Galois Theory.

V.

Solutions.

Question 210.

(R. Srinivasan, M.A.):—A curve is drawn on a generated surface cutting every generating line at an angle θ and at a distance γ from the edge of regression measured along that generator. If the radius of curvature of the edge of regression at the point where it is touched by the generator varies as $\gamma^{2} \cos^3 \theta$, find γ in terms of θ.

Solution by C. N. Srinivasiengar.

There is an evident slip in the wording of the question. By a "generated surface" is meant a "developable."

Let (a, a', a''), (b, b', b'') denote the direction cosines of the tangent and the principal normal at any point P (x, y, z) on the edge of regression and ρ its radius of curvature at P. The co-ordinates of the point Q on the generator touching the edge of regression at P and at a distance γ from P are

$$\xi = x + a\gamma ; \quad \eta = y + a'\gamma ; \quad \zeta = z + a''\gamma.$$

If s denotes the arc along the edge of regression, we have, differentiating and using the Serret-Frenet formulæ,

$$\frac{d\xi}{ds} = a + \frac{b}{\rho}\gamma + a\frac{d\gamma}{ds};$$

$$\frac{d\eta}{ds} = a' + \frac{b'}{\rho}\gamma + a'\frac{d\gamma}{ds};$$

$$\frac{d\zeta}{ds} = a'' + \frac{b''}{\rho}\gamma + a''\frac{d\gamma}{ds}.$$

The direction cosines of the tangent to the locus of Q (ξ, η, ζ) are proportional to $d\xi$, $d\eta$, $d\xi$ and are therefore given by

$$a + \frac{b}{\rho}\gamma + a\frac{d\gamma}{ds}, \text{ etc.}$$

The actual direction cosines are

$$\frac{a + \frac{b}{\rho}\gamma + a\frac{d\gamma}{ds}}{\sqrt{\left(1 + \frac{d\gamma}{ds}\right)^2 + \frac{\gamma^2}{\rho^2}}}, \text{ etc.}$$

$\therefore$ $\cos\theta$ = cosine of the angle between (a, a', a'') and the above line

$$= \frac{1+\dfrac{d\gamma}{ds}}{\sqrt{\left(1+\dfrac{d\gamma}{ds}\right)^2+\dfrac{\gamma^2}{\rho^2}}}.$$

The given relation $\rho = \gamma^n \cos^2\theta$ may be substituted, and we get the relation

$$\left(1+\frac{d\gamma}{ds}\right)^2 \sin^2\theta\cos^2\theta = \frac{1}{\gamma^{2n-2}}$$

containing γ, θ and s.

It will generally be impossible to eliminate s. In fact it is natural to expect that θ will depend upon the relation between γ and s.

Question 1435.

(K. V. VEDANTAM) :—If L, M, N are the mid-points of the sides of a triangle and D, E, F are the points of contact of the in-circle with the sides BC, CA, AB and O_1, O_2, O_3 are the ortho-centres of the triangle, AMN, BNL, CLM and O_1', O_2', O_3' are the ortho-centres of the triangles AEF, BFD, CDE; prove that

(i) O_1O_1', O_2O_2', O_3O_3' are concurrent at the Feuerbach point.

(ii) $\dfrac{O_1O_1'}{R\cos A} = \dfrac{O_2O_2'}{R\cos B} = \dfrac{O_3O_3'}{R\cos C} = \dfrac{SI}{R}$.

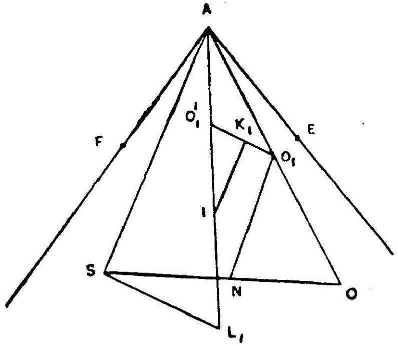
Solution by M. V. Seshadri and partial solution by S. Mahadevan.

Let O be the ortho-centre and N the nine-points centre of ABC S and I being the circum- and in-centres. It is evident that O_1, O_2, O_3 are the mid-points of AO, BO, CO.

Now since the triangle AEF is isosceles, O_1' lies on AI which is also the diameter of circle AEF. It is easy to see that

$$IA \cdot IO_1' = 2IF^2 = 2r^2. \quad \dots \quad (1)$$

Draw the line through S parallel to $O_1'O_1$ to cut AI at L_1 and IK_1 parallel to AS to cut $O_1'O_1$ at K_1.



Now $\quad AO_1 = \frac{1}{2} AO = SL = R\cos A = SA\cos A$

and $AO_1' = AI\cos A$, since AI is the diameter of circle AEF.

Hence
$$\frac{AO_1}{AO_1'} = \frac{AS}{AI}.$$

Also
$$\angle SAI = \angle O_1'AO_1$$

so that triangles SAI, O_1AO_1' are similar. ... (2)

(i) Now from the triangles SAL_1, $O_1'IK_1$ which are also similar we have
$$\frac{IK_1}{SA} = \frac{IO_1'}{AL_1} = \frac{AI\,.\,IO_1'}{AI\,.\,AL_1} = \frac{2r^2}{AI\,.\,AL_1} \text{ by (1).}$$

The triangle SIL_1 is isosceles since
$$\angle SL_1I = \angle L_1O_1'O_1 = \angle L_1IS, \text{ by (2).}$$

Hence $\quad AI\,.\,AL_1 = AS^2 - SI^2 = 2Rr.$

Therefore
$$\frac{IK_1}{R} = \frac{2r^3}{2Rr} = \frac{r}{R}.$$

Hence $IK_1 = r$, or IK_1 is the radius of the in-circle parallel to SA. Thus $O_1'O_1K_1$ joins the ends of parallel radii (IK_1 and NO_1) of the in-circle and the nine-points circle and therefore it passes through the Feuerbach point. Similarly O_2O_2', O_3O_3' also pass through the Feuer-bach point.

(ii) Next, from (2), we have

$$\frac{O_1O_1'}{AO_1} = \frac{SI}{AS}$$

i.e.

$$\frac{O_1O_1'}{R \cos A} = \frac{SI}{R} = \frac{O_2O_2'}{R \cos B} = \frac{O_3O_3'}{R \cos C}.$$

Remarks by T. R. Raghava Sastri.

The question is fully worked out in McLelland's *Geometry of the Circle*, p. 217, where O_1, O_2, O_3 are considered as the circum-centres and O_1', O_2', O_3' as the in-centres of the triangles AQR, BRP, CPQ; P, Q, R being the feet of the perpendiculars from A, B, C to the opposite sides.

Question 1439.

(K. C. SHAH, M.A.):—If $f(r)$ denotes the greatest product of n positive integers whose sum is r, show that

$$\sum_{r=kn+p}^{r=ln+q} f(r) = l^{n-q}(l+1)^{q+1} - k^{n-p+1}(k+1)^{p} - \sum_{t=k+1}^{l} t^{n}$$

where k, l, p, q are all positive integers, p and q being less than n, and the summations extend to all the integers between the limits.

Solution by T. Totadri Iyengar and R. Kothandaraman.

By an elementary theorem in inequalities, the product of n positive quantities whose sum is given is maximum when the difference between any two of them is as small as possible, preferably zero. Hence, if $0 \leqslant p < n$

$$f(mn + p) = (m+1)^{p}\, m^{n-p} = m^{n}\left(1 + \frac{1}{m}\right)^{p}.$$

Applying this equality to each of the terms on the left-hand side of the formula in the question, we get, assuming $k < l$,

$$\sum_{r=kn+p}^{r=ln+q} f(r) = k^n \sum_{s=p}^{n-1} \left(1+\frac{1}{k}\right)^s + (k+1)^n \sum_{s=0}^{n-1} \left(1+\frac{1}{k+1}\right)^s$$

$$+ (k+2)^n \sum_{s=0}^{n-1} \left(1+\frac{1}{k+2}\right)^s + \dots\dots$$

$$+ (l-1)^n \sum_{s=0}^{n-1} \left(1+\frac{1}{l-1}\right)^s + l^n \sum_{s=0}^{q} \left(1+\frac{1}{l}\right)$$

$$= k^{n+1}\left\{\left(1+\frac{1}{k}\right)^n - \left(1+\frac{1}{k}\right)^p\right\} + (k+1)^{n+1}\left\{\left(1+\frac{1}{k+1}\right)^n - 1\right\}$$

$$+ (k+2)^{n+1}\left\{\left(1+\frac{1}{k+2}\right)^n - 1\right\} + \dots\dots$$

$$+ (l-1)^{n+1}\left\{\left(1+\frac{1}{l-1}\right)^n - 1\right\} + l^{n+1}\left\{\left(1+\frac{1}{l}\right)^{q+1} - 1\right\}$$

$$= \{k(k+1)^n - k^{n-p+1}(k+1)^p\} + \{(k+1)(k+2)^n - (k+1)^{n+1}\}$$

$$+ \{(k+2)(k+3)^n - (k+2)^{n+1}\} + \dots\dots$$

$$+ \{(l-1)\,l^n - (l-1)^{n+1}\} + \{l^{n-q}(l+1)^{q+1} - l^{n+1}\}$$

$$= l^{n-q}(l+1)^{q+1} - k^{n-p+1}(k+1)^p - \sum_{t=k+1}^{l} t^n.$$

Question 1444.

(SELECTED):—Prove that

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \dots = \frac{\theta^2}{1^3} + \frac{\theta^4}{2^3} + \frac{\theta^6}{3^3} + \dots$$

where $$\theta = 2\sin\frac{\pi}{10}.$$

Remarks by S. D. Chowla.

The question as it stands is wrong, as each term of the series on the left is greater than the corresponding term on the right. It is stated in this form in Edward's *Integral Calculus*, Vol. II, page 291, Q. 41. iv. The question is there referred to Landen's *Math. M*

195325

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1547. (V. RAMASWAMI AIYAR, M.A.) :—Show that the number of ways of connecting n villages with one another by $n-1$ roads, each of them connecting two of the villages directly, is n^{n-2}.

[*A Prize Problem*, see Editor's note, p. *89*].

1548. (S. NARAYANA AIYAR, M.A.) :—Establish the following relations :—

(1) $$\frac{\operatorname{sn} 3u}{\operatorname{sn} u \operatorname{cn} 2u + \operatorname{sn} 2u \operatorname{cn} u} = \frac{1 + \operatorname{dn} 3u}{\operatorname{dn} u + \operatorname{dn} 2u},$$

(2) $$\frac{\operatorname{sn} 3u}{\operatorname{sn} u \operatorname{dn} 2u + \operatorname{sn} 2u \operatorname{dn} u} = \frac{1 + \operatorname{cn} 3u}{\operatorname{cn} u + \operatorname{cn} 2u}.$$

1549. (S. SIVASANKARANARAYANA PILLAI) :—

If $$\phi(mn) = \sum_{l=1}^{\infty} d^{t} f\left(\frac{m}{d^r}\right) f\left(\frac{n}{d^s}\right)$$

then $$f(m) f(n) = \sum_{d=1}^{\infty} \mu(d) d \; \phi\left(\frac{m}{d^{r+s}}\right)$$

where $\phi(x)$ and $f(x)$ are arithmetical functions.

1550. (PROF. K. J. SANJANA) :—Let n be an odd prime and N denote the number $2^n - 1$.

(1) When N is also prime, prove that $N - 1$ is a multiple of $2n$.

(This result follows directly from Fermat's Theorem).

(2) When N is the product of two factors A and B, prove that $A \sim B$ is a multiple of $2n$.

1551. (C. N. SRINIVASIENGAR) :—(1) The parabolic curve of any skew ruled surface is the envelope of its non rectilinear asymptotic lines.

(2) Every skew ruled surface (other than a quadric) possesses a finite number of plane lines of curvature given by the several generators comprising the parabolic curve.

UNSOLVED QUESTIONS FOR WHICH SOLUTIONS ARE INVITED.

256 (T. SUNDARA RAO, B.A.):—Show how to describe cyclic quadrilaterals whose sides, diagonals, radius of circum-circle and area are integers.

258. (T. SUNDARA RAO, B.A.) :—ABCD is a quadrilateral. A, B are fixed pivots (fulcra) and C, D are movable points. The quadrilateral is deformed and changes shape. Discuss the curves traced by points of CD and by points outside but rigidly connected with CD.

Obituary.

T. P. TRIVEDI.

TRIBHOVAN P. TRIVEDI, who was Professor of Mathematics in the D. J. Sind College, Karachi, died in April 1929 at the age of 41.

After a brilliant academic career in the Samaldas College, Bhavnagar, where he came under the influence of Prof. K. J. Sanjana, and in the Elpinstone College, Trivedi was appointed Chief Professor of Mathematics in St. John's College, Agra, early in 1910. Three years later he was transferred to the Samaldas College, and in 1915 secured the Chief Professorship of Mathematics in the D. J. S. College, Karachi.

Trivedi has been a member of the Indian Mathematical Society from 1913, and is the joint author with Prof. Sanjana of several questions and solutions which appeared in this Journal.

Besides his mathematical activities, Trivedi took considerable interest in the improvement of Guzarati literature, and was the founder and one of the most active members of the Guzarati Mandal.

LIST OF

PERIODICALS AND BOOKS RECEIVED IN THE LIBRARY

during the months of June and July 1929.

1 Abhandlungen aus dem Mathematichen Seminar der Hamburgischen Universitat, **7**, 1.

2 Academie des Sciences, **3**, 2, 3, **4**, 1, **7**, 4, **8**, 1, **9**, 3, **10**, 2, 3 **11**, 1, 2, **12**, 11, 2.

3 American Mathematical Monthly, **36**, 4 (3 copies). **36**, 5 (3 copies).

4 Annales de l'Ecole Normale Superieure, **64**, 1, 2.

5 Annals of Mathematics, **30**, 2.

6 Astrophysical Journal, **69**, 2, 3.

7 Bulletin des Sciences Mathematiques, **53** Feb., March, April and May.

8 Bulletin del Seminaria Mathematico Argentino, **1**, 8.

9 Crelle's Journal, **160**, 3.

10 Mathematical Gazette, **14**, 200 (3 copies): 201 (3 copies).

11 Mathematische Annalen, **101**, 4.

12 Philosophical Magazine. **7**, 46, 47.

13 Philosophical Transactions of the Royal Society, **228**, pp. 197—234, 235—274, 275—329.

14 Popular Astronomy, **37**, 5, 6.

15 Proceedings of the Cambridge Philosophical Society, **25**, 2.

16 Proceedings of the London Mathematical Society, **29**, 5.

17 Proceedings of the Physico-Mathematical Society of Japan. **11**, 1 2, 3, 4, 5, 6 (2 copies).

18 Proceedings of the Royal Society of London, **124**, 793, (2 copies) 794. (2 copies). 795.

19 Publication de la Faculte des Sciences de L'University Masaryk, 1928, 104, 1929, 106, 107, 108.

20 Transactions of the Royal Society of South Africa, **17**, 3, 4.

21 Wiskundige Opgaven met de Oplossingen, 1929.

Books and Calendars received in presentation.

1 The Theory of Restricted Relativity, by Domenico Ferroli.

2 Theorems stated by Ramanujan.

Approximations connected with e^x: V, by G. N. Watson.

Theorems on Continued Fractions: VI, by G. N. Watson.

Theorems on Continued Fractions: VII, by G. N. Watson.

3 A New proof of the Rogers-Ramanujan Identities, by G. N. Watson.

4 Art and Literature including a Selection from the Library of Sir Edmund Gosse.

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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